

Criticality in unbounded-types branching processes

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Abstract

Conditions for almost sure extinction are studied in discrete time branching processes with an infinite number of types. It is not assumed that the expected number of children is a bounded function of the parent's type. There might also be no integer m such that there is a lower positive bound, uniform over the ancestor's type, for the probability that a population is extinct at the m -th generation. A weaker condition than the existence of such an m is seen to lead to extinction almost surely if the sequence of expected generation sizes does not tend to infinity. Some criteria for a positive probability of nonextinction are given. Examples are provided by extending to our setting two applications, namely Leslie population dynamics and processes arising in continuum percolation in which the offsprings follow Poisson point distributions.

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1 Introduction

We study the subject of giving conditions for the extinction probability to be 1 in discrete time branching processes with an infinite number of types. Known results suppose that the expected number of children is a bounded function of the type of their parent (Harris, 1963; Mode, 1971). Here we do not prevent types from being arbitrarily large. This means that there might be no uniform bound over the types for the expected number of children, nor a number m of generations such that there is a uniform positive lower bound for the probability that an individual will have no descent at the m -th generation. A weaker condition than the latter is introduced in Section 2 and it is shown to be sufficient for almost sure extinction if the sequence of expected generation sizes does not tend to infinity, even if these expectations fluctuate without an upper bound. Existence of a strictly dominant eigenvalue of the integral operator associated to the offspring means is not assumed in our propositions. Focusing on the expected number of individuals in subsets of types that are in some sense larger

than the type of the initial individual, criteria for the extinction probability to lie below 1 are obtained in Section 3. Section 4 provides examples by extending to unbounded types a couple of well known applications. The possibility of constructing counterexamples shows that the sufficient condition for almost sure extinction in Section 2 is not always necessary and leaves a criticality problem open.

2 Almost sure extinction

Let Z_0, Z_1, \dots be a discrete-time branching process with types in a space X , countable or not. In the uncountable case, X is an Euclidean space and Z_0, Z_1, \dots a sequence of point distributions that follows the branching process definition as given in Harris (1963), Chapter 3. The number of individuals with type in a measurable set $A \subset X$ at the n -th generation will be denoted by $Z_n(A)$. The symbols $P_x(\cdot)$, E_x and q_x will stand for probabilities, expectations and the extinction probability if the process is initiated with a single individual of type $x \in X$.

Condition 1 For any number of children $k > 0$, there exist a number of generations $m(k)$ and a real number $q(k) > 0$ such that

$$\inf_{x \in X} P_x(Z_{m(k)}(X) = 0 | 1 \leq Z_1(X) \leq k) \geq q(k).$$

Proposition 2 If Condition 1 holds and $\liminf_n E_x Z_n(X) < \infty$, then $q_x = 1$.

Proof. By Fatou's Lemma, $\liminf_n E_x Z_n(X) < \infty$ implies $E_x \liminf_n Z_n(X) < \infty$, i.e. $\sum_{k \geq 0} P_x(\liminf_n Z_n(X) > k) < \infty$. Thus given $\epsilon > 0$, there is a positive integer k_ϵ such that $P_x(\liminf_n Z_n(X) > k_\epsilon) \leq \epsilon$.

So $P_x(Z_n(X) > k_\epsilon \text{ for all but finite } n\text{'s}) \leq \epsilon$ and therefore

$$P_x(Z_n(X) \leq k_\epsilon, i.o.) > 1 - \epsilon.$$

We shall see that for any positive integer k , $P_x(1 \leq Z_n(X) \leq k, i.o.) = 0$. This will complete the proof because then for all ϵ , $P_x(Z_n(X) = 0, i.o.) > 1 - \epsilon$ and the extinction probability must equal 1. Let

$$N_0(k) = 0$$

and for $i = 1, 2, \dots$,

$$N_i(k) = \min\{n > N_{i-1}(k) - 1 + m(k) : 1 \leq Z_n(X) \leq k\}$$

with $\min \emptyset = \infty$. Now,

$$P_x(1 \leq Z_n(X) \leq k, i.o.) = P_x(\cap_i \{N_i(k) < \infty\}) = \lim_{i \rightarrow \infty} P_x(N_i(k) < \infty),$$

so it is enough to prove that $P_x(N_i(k) < \infty) \leq (1 - q(k)^k)^i$.

For $i = 0$ we have

$$P_x(N_0(k) < \infty) = 1 = (1 - q(k)^k)^0$$

and for $i > 0$

$$\begin{aligned} P_x(N_{i+1}(k) < \infty) &\leq P_x(N_i(k) < \infty, Z_{N_i(k)-1+m(k)}(X) > 0) \\ &= \sum_{n=1}^{\infty} P_x(Z_{n-1+m(k)}(X) > 0 | N_i(k) = n) P_x(N_i(k) = n). \end{aligned}$$

But given $N_i(k) = n$, there are at most k individuals at generation n . These must be the children of at most k individuals out of generation $n-1$, which have at most k children each. Hence by Condition 1, $1 - q(k)^k$ is an upper bound for $P_x(Z_{n-1+m(k)}(X) > 0 | N_i(k) = n)$ yielding

$$\begin{aligned} P_x(N_{i+1}(k) < \infty) &\leq (1 - q(k)^k) \sum_{n=1}^{\infty} P_x(N_i(k) = n) \\ &= (1 - q(k)^k) P_x(N_i(k) < \infty). \end{aligned}$$

The desired result follows by induction.

3 Positive probability of nonextinction

Let Z_0, Z_1, \dots be a discrete-time branching process with types in a countable or uncountable set X as in the preceding section. Given a type x and a positive integer n , by the n -th generation descendants of x we will mean the individuals in the n -th generation of a population that begins with a single individual of type x . Let us consider the following order relation in the set of types. Given two types x and y we shall say that x is larger than y if for every measurable subset A of types and any n ($n = 1, 2, \dots$), the number of n -th generation descendants with types in A is stochastically larger for x than for y .

Proposition 3 Let y be a type in X . Suppose that there exist a measurable set A whose elements are larger than y and a positive integer i such that $E_y Z_i(A) > 1$. Then for any x larger than y we have $q_x < 1$.

Proof. Let S_0, S_1, \dots be a single-type branching process defined by

$$S_0 = 1$$

and for $k = 0, 1, \dots$,

$$P(S_1 = k) = P_y(Z_i(A) = k).$$

As $E_y Z_i(A) > 1$, this process is supercritical. Now consider Z_0, Z_i, Z_{2i}, \dots , the multitype process at those generations that are multiples of i , beginning with a

single individual whose type is x . Since x and all elements of A are larger than y , $Z_{ni}(A)$ is stochastically larger than S_n for any n . As $Z_{ni}(X) \geq Z_{ni}(A)$, we have $q_x < 1$.

Corollary 4 If for a type x there exists i such that $E_x Z_i(\{x\}) > 1$ then $q_x < 1$.

Corollary 5 Suppose that there exist y , i and A as in the proposition. If there is a type x , an integer j and a set $B \subset X$ whose elements are larger than y , such that $E_x Z_j(B) > 0$, then $q_x < 1$.

Corollary 6 If there is a smallest type s in the sense that every type x is larger than s , and $E_s Z_i(X) > 1$ for some i , then $q_x < 1$ for every x .

4 Examples

Before going into the examples it should be remarked that in all the processes we are dealing with, the expectations $E_x Z_n(A)$ verify

$$E_x Z_{n+1}(A) = \int_X E_y Z_n(A) dE_x Z_1(y),$$

for $n = 1, 2, \dots$, as proved in Harris (1963), Chapter 3, and in Mode (1971), Chapter 6. The set A is any measurable subset of X and $dE_x Z_1(y)$ means that the integration is performed with respect to the measure defined by $E_x Z_1(S)$ for any measurable subset S of X . Under discrete interpretation, the right hand side of the equality becomes the result of rising the first generation mean matrix to the $(n+1)$ -th power and taking the sum of the elements in the row of x that lie in the columns that correspond to types in A . In the continuous case, suppose that $k(x, y)$ is a nucleus such that for any measurable $A \subset X$ we have $E_x Z_1(A) = \int_A k(x, y) d\mu(y)$ for some measure μ on X . Then the formula yields the iteration of the integral operator K associated to k and μ , allowing us to write $E_x Z_{n+1}(A) = \int_A K^n(k(x, y)) d\mu(y)$.

In all our examples we shall both have

$$\sup_{x \in X} E_x Z_1(X) = \infty$$

as

$$\inf_{x \in X} P_x(Z_m(X) = 0) = 0$$

for any m . That the expectations are not bounded will be evident. In order to prove the second property, the following proposition will be useful.

Proposition 7 Suppose that for any positive integer h , $\inf_{x \in X} P_x(Z_1(X) \leq h) = 0$ and that there exists $\alpha > 0$ such that for every

$x \in X$, $P_x(Z_1(X) > 0) \geq \alpha$. Then $\inf_{x \in X} P_x(Z_m(X) = 0) = 0$ for any number of generations m .

Proof. Let $\{x_h : h = 1, 2, \dots\} \subset X$ be such that $P_{x_h}(Z_1(X) \leq h) \rightarrow 0$ when $h \rightarrow \infty$. For any m ,

$$\begin{aligned} & P_{x_h}(Z_m(X) = 0) \\ & \leq P_{x_h}(Z_1(X) \leq h) + P_{x_h}(Z_m(X) = 0 | Z_1(X) > h) P_{x_h}(Z_1(X) > h) \\ & \leq P_{x_h}(Z_1(X) \leq h) + (1 - \alpha^{m-1})^h. \end{aligned}$$

So $\liminf_h P_{x_h}(Z_m(X) = 0) \leq 0$ and the proposition is proved.

4.1 Leslie dynamics

We shall consider the infinite types analog of the finite-types branching process studied by Pollard (1966), which is the stochastic version of the deterministic population dynamics described by Leslie (1945). The set of types will be here $\{x_0, x_1, \dots\}$. If we adopt the interpretation of types as age classes, then the individuals of type x_0 are the youngest ones. An x_0 may bear individuals of type x_0 or may produce an individual of type x_1 by surviving enough time to enter in the next age class. In general x_i ($i = 0, 1, \dots$) may only produce x_0 's and eventually an x_{i+1} , so that the infinite mean matrix has only two positive entries at each row, $E_{x_i}Z_1(\{x_0\})$ and $E_{x_i}Z_1(\{x_{i+1}\})$. For simplicity, both the individuals x_0 as the x_{i+1} arising from an x_i will be called its children.

Our examples will have first generation means $E_{x_i}Z_1(\{x_0\}) = p^{1-i}$ and $E_{x_i}Z_1(\{x_{i+1}\}) = p^2$ for p in $(0, 1)$, so the infinite mean matrix is

$$\begin{pmatrix} p & p^2 & 0 & 0 & 0 & \dots \\ 1 & 0 & p^2 & 0 & 0 & \\ p^{-1} & 0 & 0 & p^2 & 0 & \\ p^{-2} & 0 & 0 & 0 & p^2 & \\ \dots & & & & & \dots \end{pmatrix}.$$

By what we noticed at the beginning of this section, the first row of the n -th power of this matrix consists of the n -th generation means $E_{x_0}Z_n(\{x_0\})$, $E_{x_0}Z_n(\{x_1\})$, It can be seen by induction over n that these entries are

$$E_{x_0}Z_n(\{x_i\}) = \begin{cases} (1/2)(2p)^n(p/2)^i & \text{if } 0 \leq i < n \\ p^{2n} & \text{if } i = n \\ 0 & \text{if } i > n \end{cases}.$$

In particular, $E_{x_0}Z_n(\{x_0\}) = (1/2)(2p)^n$ so by Corollary 4, $q_{x_0} < 1$ if $p > 1/2$. By a simple case of Corollary 5, $q_{x_i} < 1$ for any i .

In case $p \leq 1/2$ the mean number of individuals of any type at the n -th generation ($n = 1, 2, \dots$) in a population that started with a single individual of

type x_0 is bounded by a constant. Indeed

$$E_{x_0} Z_n(X) = \sum_{i=0}^n E_{x_0} Z_n(\{x_i\}) = (1/2)(2p)^n \frac{1 - (p/2)^n}{1 - p/2} + p^{2n} < 1.$$

Thus $\liminf_n E_{x_0} Z_n(X) < \infty$ is satisfied. But if $p < 1/2$ we can bound the expected total number of descendants in all generations because

$$\sum_{n=1}^{\infty} E_{x_0} Z_n(X) < \frac{1}{2-p} \sum_{n=1}^{\infty} (2p)^n + \sum_{n=1}^{\infty} p^{2n} < \infty.$$

So $E_{x_0} \sum_{n=1}^{\infty} Z_n(X) < \infty$ and therefore $q_{x_0} = 1$.

We shall now consider two different offspring distributions behind our means. In both examples, the number of x_0 -type children of an x_i parent ($i = 0, 1, \dots$) will follow a Poisson distribution with mean p^{1-i} and the number of x_{i+1} -type children will be 1 with probability p^2 and 0 with probability $1 - p^2$. The difference will lie in the joint distribution for these marginals, as described below. It can already be seen that $\inf_{x \in X} P_x(Z_m(X) = 0) = 0$ for any m because having these marginal distributions implies the hypotheses of Proposition 7.

The first choice of distributions for the children of the x_i 's ($i = 0, 1, \dots$) is one that satisfies Condition 1. We take for each x_i the joint distribution that makes the number of children of type x_0 independent of the number of x_{i+1} children. This family satisfies Condition 1 for $m(k) = 2$ because

$$\begin{aligned} & P_{x_i}(Z_2(X) = 0 | 1 \leq Z_1(X) \leq k) > \\ & > P_{x_i}(Z_2(X) = 0 | Z_1(\{x_{i+1}\}) = 0, 1 \leq Z_1(X) \leq k) \cdot \\ & \cdot P_{x_i}(Z_1(\{x_{i+1}\}) = 0 | 1 \leq Z_1(X) \leq k) \end{aligned}$$

where

$$P_{x_i}(Z_2(X) = 0 | Z_1(\{x_{i+1}\}) = 0, 1 \leq Z_1(X) \leq k) \geq (P_{x_0}(Z_1(X) = 0))^k,$$

while for any integer h between 1 and k ,

$$\begin{aligned} & P_{x_i}(Z_1(\{x_{i+1}\}) = 0 | Z_1(X) = h) = \\ & = \frac{(1 - p^2)e^{-p^{1-i}}(p^{1-i})^h/h!}{(1 - p^2)e^{-p^{1-i}}(p^{1-i})^h/h! + p^2e^{-p^{1-i}}(p^{1-i})^{h-1}/(h-1)!} > \frac{1}{1+k}. \end{aligned}$$

Thus

$$\inf_{x_i \in X} P_{x_i}(Z_2(X) = 0 | 1 \leq Z_1(X) \leq k) > (P_{x_0}(Z_1(X) = 0))^k \frac{1}{1+k}.$$

Since it has been shown that $\liminf_n E_{x_0} Z_n(X) < \infty$ for $p \leq 1/2$, we have $q_{x_0} = 1$ by Proposition 2.

The second family of distributions does not satisfy Condition 1 because there is enough positive correlation between bearing few children and having one very prolific among them. A way to achieve this is by requiring that at least for $k = 1$ and for infinite i 's,

$$P_{x_i}(Z_1(\{x_{i+1}\}) = 1 | 1 \leq Z_1(X) \leq k) = 1.$$

This is possible because due to the marginal distributions $P_{x_i}(Z_1(X) \leq 1) \rightarrow 0$ when $i \rightarrow \infty$. So for any i greater than some \tilde{i} , $P_{x_i}(Z_1(X) \leq 1) < p^2$. Since p^2 is the probability of having one x_{i+1} , we can define for those i 's a joint distribution that loads all of $P_{x_i}(Z_1(X) \leq 1)$ only on the probability of having no x_0 and one x_{i+1} . So for $i > \tilde{i}$, $P_{x_i}(Z_1(\{x_{i+1}\}) = 1 | Z_1(X) = 1) = 1$, as we required. Now we are satisfying an analogous hypothesis as that of Proposition 7, namely that for any integer h , $\inf_{x_i \in X} P_{x_i}(Z_2(X) \leq h | Z_1(X) = 1) = 0$. By this and the fact that there exists $\alpha > 0$ such that for every x_i , $P_{x_i}(Z_1(X) > 0) > \alpha$, an analogous proof as that of Proposition 7 gives

$$\inf_{x_i \in X} P_{x_i}(Z_m(X) = 0 | Z_1(X) = 1) = 0$$

for any number of generations m . This means that Condition 1 is not satisfied because it fails for $k = 1$. Since it was already proved that $q_{x_0} = 1$ for $p < 1/2$, a consequence of this example is that the hypotheses of Proposition 2 are not necessary for the extinction probability to be 1. For $p = 1/2$, the criticality problem remains here unsolved.

Finally it can be said about the case $p \leq 1/2$ for $i = 1, 2, \dots$, that $q_{x_i} = 1$ if $q_{x_0} = 1$ by types communication. In fact, $q_{x_i} < 1$ implies $q_{x_{i-1}} < 1$.

4.2 A non recurrent variant

The following modification of the unbounded-types Leslie scheme provides an example of the application of Proposition 2 in a non recurrent case. Let $X = \{x_0, x_1, \dots\}$, $p \in (0, 1)$ and the infinite mean matrix be

$$\begin{pmatrix} p & 0 & 0 & 0 & 0 & \dots \\ 1 & 0 & p & 0 & 0 & \\ p^{-1} & 0 & 0 & p & 0 & \\ p^{-2} & 0 & 0 & 0 & p & \\ \dots & & & & & \dots \end{pmatrix},$$

i.e. the only positive expectations are $E_{x_i}Z_1(\{x_0\}) = p^{1-i}$ for $i = 0, 1, \dots$ and $E_{x_i}Z_1(\{x_{i+1}\}) = p$ for $i = 1, 2, \dots$. In a population that begins with a single individual of type x_1 , only types x_0 and x_{n+1} can be present at the n -th generation and it can easily be seen that

$$E_{x_1}Z_n(\{x_0\}) = \sum_{i=0}^{n-1} p^i$$

and

$$E_{x_1} Z_n(\{x_{n+1}\}) = p^n.$$

Hence

$$E_{x_1} Z_n(X) = \sum_{i=0}^n p^i < \frac{1}{1-p}.$$

In the same way as in the preceding examples, we can find a family of distributions with the given means that make $q_{x_1} = 1$ by Proposition 2, and another family of distributions can be defined, such that the criticality problem is not solved by Proposition 2. Anyway we notice that although the line of descent x_i, x_{i+1}, \dots ($i \geq 1$) is a source of growing quantities of individuals of type x_0 , it is clearly once broken with probability 1. So as $q_{x_0} = 1$ we have $q_{x_i} = 1$ for any $i \geq 1$.

4.3 Poisson process descent

Continuum percolation clusters in R^d can be dominated by coupling with branching processes so that extinction in the branching process implies that the percolation cluster is finite. This is done by defining a parent-child relation between objects that intersect each other in the percolation setting. Extra individuals are eventually added to the families so defined, so that the resulting families follow the independent offspring distributions of a branching process. This method appears in Hall (1985) and it is used and referenced in Meester and Roy (1996). It is a way to obtain an upper bound for λ , the Poisson intensity parameter of the objects' centers, such that below this bound the finiteness of the clusters is almost sure.

This application motivates us to analyze the criticality subject in branching processes that would dominate continuum percolation clumps of objects whose size might be arbitrarily large. We shall consider the case of spheres in R^d ($d \geq 1$) with not uniformly bounded random radii, which will be the types in the branching process. Given a sphere in R^d , its offspring is determined by spheres that have centers in a homogeneous Poisson process in R^d with intensity λ and have iid radii. The children are those spheres that intersect the given sphere. The offspring of each of these children is generated by a new independent Poisson process and independent radii and so on. It can be checked that we both have $\sup_{x \in X} E_x Z_1(X) = \infty$ as $\inf_{x \in X} P_x(Z_m(X) = 0) = 0$ for any m by Proposition 7.

Let F be the cumulative distribution function of the radii and assume that it has an inverse $F^{-1} : [0, 1) \rightarrow [0, \infty)$ when restricted to $[0, \infty)$. A way to deduce the expression of a mean nucleus is to realize children spheres as homogenous Poissonian points with intensity λ in $R^d \times [0, 1)$. The first d dimensions give the center of the child sphere and the last one is the image of its radius by F . Let $u \in [0, 1)$ be a value for the last component of a point in our representation. We know that spheres with radius $F^{-1}(u)$ intersect a parent sphere of radius x if

their centers lie inside the parent sphere or outside of it but at an R^d distance within $(x, x + F^{-1}(u)]$ to the parent's center. So in our $(d + 1)$ -dimensional representation, the points that correspond to $F^{-1}(u)$ -type children of an x lie in the d -dimensional sphere that has the same center components as the center of the parent sphere, radius $x + F^{-1}(u)$ and last component u . For any $x_1 \in (0, \infty)$, the union of such spheres over u in the interval $[F(0), F(x_1)]$ is a solid with volume

$$\int_{F(0)}^{F(x_1)} v_d(x + F^{-1}(u))^d du = \int_0^{x_1} v_d(x + y)^d dF(y),$$

where v_d is the volume of the unit sphere in R^d . Thus we obtained the volume of the subset of R^{d+1} whose points represent those children of an x that have last component $u \in [F(0), F(x_1)]$ or equivalently, type in $[0, x_1]$. As our Poisson process is homogeneous with parameter λ , the expected number of children of an x with types in $[0, x_1]$ is

$$E_x Z_1([0, x_1]) = \lambda \int_0^{x_1} v_d(x + y)^d dF(y).$$

Let $k_d(x, y) = \lambda v_d(x + y)^d$. If we suppose that the distribution of the radii has finite $2d$ moment (i.e. sphere content has finite variance) then the integral operator $K_d : L^2([0, \infty), dF) \rightarrow L^2([0, \infty), dF)$ defined by k_d is compact because k_d is a Hilbert-Schmidt nucleus. Indeed, $\int_0^\infty \int_0^\infty k_d^2(x, y) dF(x) dF(y) < \infty$. Since K_d is also strictly positive, it has a dominant eigenvalue $\rho > 0$ and therefore by $E_x Z_n([0, \infty)) = \int_0^\infty K_d^{n-1}(k_d(x, y)) dF(y)$, the sequence $\{E_x Z_n([0, \infty)) : n = 1, 2, \dots\}$ is bounded if and only if $\rho \leq 1$, for any initial type x .

It will now be seen that for any dimension d and every type x , we have $\rho \leq 1$ if and only if $q_x = 1$. To complete the hypotheses of Proposition 2 when $\rho \leq 1$ it only remains to show that Condition 1 is satisfied. Fixing $\tilde{x} > 0$, we shall first prove that there is a positive bound for the probability of having all children with types below \tilde{x} , or more precisely that for any $k \geq 1$,

$$\inf_{x \in [0, \infty)} P_x(Z_1([0, \infty)) = Z_1([0, \tilde{x}]) + 1 \leq Z_1([0, \infty)) \leq k) > b(k)$$

for some $b(k) > 0$. By the representation of children as homogeneous Poissonian points, we see that given a number of children, each choice of type for a child is independent of the others and the probability that the choice falls below \tilde{x} when the parent type is x , equals the volume ratio

$$\frac{\int_0^{\tilde{x}} v_d(x + y)^d dF(y)}{\int_0^\infty v_d(x + y)^d dF(y)} = \frac{x^d \int_0^{\tilde{x}} dF(y) + \sum_{i=1}^d \binom{d}{i} x^{d-i} \int_0^{\tilde{x}} y^i dF(y)}{x^d + \sum_{i=1}^d \binom{d}{i} x^{d-i} \int_0^\infty y^i dF(y)}.$$

The integrals are finite by the assumption about the moments, so this probability is a continuous and strictly positive function of x . Letting $x \rightarrow \infty$ gives the strictly positive value $\int_0^{\tilde{x}} dF(y)$. Hence the function has a positive lower bound

β_x and we can take $b(k) = \beta_x^k$. Finally, since an individual whose type is in $[0, \tilde{x}]$ has no children with a probability larger or equal than that for \tilde{x} , Condition 1 is satisfied by

$$\inf_{x \in [0, \infty)} P_x(Z_2([0, \infty)) = 0 | 1 \leq Z_1([0, \infty)) \leq k) \geq \beta_x^k (P_x(Z_1(X) = 0))^k.$$

If $\rho > 1$, taking $x = 0$ as the initial type, $E_0 Z_i([0, \infty)) > 1$ for some generation i . By Corollary 6, we have $q_x < 1$ for every $x \in [0, \infty)$ if we prove that 0 is a smallest type. In fact, $x \geq y$ implies that x is larger than y in the sense of Section 3. This assertion can be justified by coupling, constructing a process that begins with an individual of type y jointly with one that begins with an x . We first assign children to the y by placing points with intensity λ in the solid of R^{d+1} that corresponds to y 's first generation descent, i.e. the union over $u \in [0, 1)$ of the R^d spheres of radius $y + F^{-1}(u)$. This set is contained in the solid corresponding to the first generation descent if the parent's type were x because $y + F^{-1}(u) \leq x + F^{-1}(u)$. So we can assign to x the same children as those of y plus eventually some more in order to follow the children distribution of an x parent. In the subsequent generations, x receives the same descent as that of y plus the descent of those first generation extra children. This makes every n -th generation at any $A \subset X$, equal or larger for x than for y .

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